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IMPES Stability: The CFL Limit

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Abstract

This work considers cocurrent, three-dimensional, single-phase miscible and two-phase immiscible, hyperbolic flow in a general grid, structured or unstructured. A given grid block or control volume may have any number of neighbors. Heterogeneity, anisotropy, and viscous and gravity forces are included, while tensor considerations are neglected. The flow equations are discretized in space and time, with explicit composition and mobility used in the interblock flow terms (the Impes case).

Published stability analyses for this flow in a less general framework indicate that the CFL number must be < 1 or < 2 for stability. A recent paper reported non-oscillatory stability of one- and two-dimensional Buckley-Leverett two-phase simulations for CFL < 2. A subsequent paper claimed to predict this CFL < 2 limit from a stability analysis. This work gives a different reason for that stability up to CFL < 2.

This work shows that the eigenvalues of the stability matrix are equal to its diagonal entries, for any ordering scheme. The eigenvalues are in turn equal to 1-CFL$_i$, which leads to a conclusion of an early paper that CFL < 1 is required for non-oscillatory stability. CFL values between 1 and 2 give oscillatory stability. In general, our Impes simulations require the non-oscillatory stability ensured by CFL < 1.

1. Introduction

The Impes formulation[1-3] treats interblock flow rates implicitly in pressure, but explicitly in saturations and compositions. This explicit treatment gives rise to a conditional stability,

$$\frac{F_i \Delta t}{V_{pi}} < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

F_i is some function of rates and/or reservoir and fluid properties associated with grid block i and Δt is maximum stable time step. The terms CFL$_i$ and CFL are defined as $\text{CFL}_i = F_i \Delta t / V_{pi}$ and $\text{CFL} = Max(i)\,\text{CFL}_i$.

The Appendix gives a brief derivation of the well known explicit difference equation

$$S_{i,n+1} = D_i S_{i+1,n} + (1 - 2D_i - C_i) S_{i,n} + (D_i + C_i) S_{i-1,n} \quad \ldots \quad (2)$$

which describes one-dimensional (1D) two-phase flow. For gas-oil flow, D_i is $(T P'_{cgo} \Psi)_i \Delta t / V_{pi}$. C_i is $q f'_{gi} \Delta t / V_{pi}$ or CFL$_i$ when P_{cgo} is 0.

Prior to 1950, mathematicians developed stability analyses for Eq. 2. Subsequent work used their methods and results to derive stability conditions for Impes.[4-11] In 1968 [4] the following stability conditions were derived for 1D, 2D, or 3D flow:

$$\frac{\Delta t}{V_p} 2 \Psi P'_{cgo} (T_x + T_y + T_z) < 1 \qquad \text{if } C_i = 0 \ \ldots\ldots\ldots \ (3)$$

$$\frac{\Delta t}{V_p} f'_g (|q_x| + |q_y| + |q_z|) < 1 \qquad \text{if } D_i = 0 \ \ldots\ldots\ldots \ (4)$$

Those conditions and the additional following result, when both D_i and C_i are nonzero, were derived by Todd *et al* [5] in 1972 and Russell[8] in 1989:

$$\frac{\Delta t}{V_p} [2 P'_{cgo} \Psi (T_x + T_y + T_z) + f'_g (|q_x| + |q_y| + |q_z|)] < 1 \quad (5)$$

Todd *et al* stated their Condition 5 applied if the total flow rates q_x, q_y, q_z were everywhere positive. Watts and Rame[11] present an analysis which they say predicts Young and Russell's[10] experimental observation of a limit < 2 in Condition 4. This paper mainly deals with the value of that limit or reasons for its variability.

Section 2 expresses the cocurrent hyperbolic flow equations in a general grid in a simple fashion. Section 3 discusses the well known fact that the stability limit for this cocurrent hyperbolic flow is CFL < 1 or CFL < 2, depending upon the definition of the term "stability". Section 4 shows the desirability of CFL < 1 for the type of stability required in our reservoir problems. Section 5 gives a simple explanation for the stability limit CFL < 2 in Buckley Leverett problems.

2. Hyperbolic Flow in a General Grid

We consider cocurrent two-phase hyperbolic flow in a general grid, structured or unstructured. Let q_{ij} denote the absolute value of the total flow rate between block i and one of its neighbors j. The corresponding flow of the conserved component is $q_{ij} f_{ij}$ where fractional flow f_{ij} is the fraction of the component in the flowing stream. Let $S\,V_{pi}$ denote mass of the conserved component in a grid block. For a two-phase water-oil case, q_{ij} is volume of water+oil/day, f_{ij} is q_{wij}/q_{ij}, and S is water saturation. For the miscible flow case, q_{ij} is total mols/day, f_{ij} is S, and S is mol fraction of the component. For hyperbolic flow, by the upstream principle, f_{ij} is a function of upstream S, S_i or S_j. The double subscript on f_{ij} is necessary because of the gravity component of f_{ij} (see Eq. A-6).

We can always number the grid blocks so that for each block i, all upstream neighbor blocks have indices $j < i$ and all downstream neighbor blocks have indices $j > i$. This simplifies notation. The resulting error amplification matrix is lower triangular and its eigenvalues are equal to its diagonal elements.[12] The mass conservation equations at grid block i for total mass and the conserved component are

$$\sum_{j=1}^{j=i-1} q_{ij} = \sum_{j=i+1}^{j=N} q_{ij} = q_i \quad \ldots\ldots (6)$$

$$\frac{V_{pi}}{\Delta t}(S_{i,n+1} - S_{i,n}) = \sum_{j=1}^{j=i-1} q_{ij} f_{ij}(S_{j,n}) - \sum_{j=i+1}^{j=N} q_{ij} f_{ij}(S_{i,n}) \quad . \ (7)$$

where q_i denotes the total flow rate into and out of block i. The f_{ij} values are dated at time level n and are known. All q_{ij}, dated at time n+1, are known from an Impes solution. Each q_{ij} is positive if block j is a neighbor of block i, 0 if not.

3. Stability Analysis

The usual method of deriving an error equation for stability analysis is as follows. If $S^*_{i,n}$ is the exact solution of Eq. 7 then both $S_{i,n}$ and $S^*_{i,n}$ satisfy it. Writing Eq. 7 once using $S_{i,n}$, again using $S^*_{i,n}$, and subtracting shows that the error $\varepsilon_{i,n} = S_{i,n} - S^*_{i,n}$ satisfies

$$\frac{V_{pi}}{\Delta t}(\varepsilon_{i,n+1} - \varepsilon_{i,n}) = \sum_{j=1}^{j=i-1} q_{ij}(f_{ij}(S_{j,n}) - f_{ij}(S^*_{j,n}))$$

$$- \sum_{j=i+1}^{j=N} q_{ij}(f_{ij}(S_{i,n}) - f_{ij}(S^*_{i,n})) \quad \ldots . (8)$$

Using Taylor series, terms of type $f(S) - f(S^*)$ become $f'\varepsilon$, and rearranging gives

$$\varepsilon_{i,n+1} = \sum_{j=1}^{j=i-1} c_{ij}\varepsilon_{j,n} + (1 - c_{ii})\varepsilon_{i,n} = \sum_{j=1}^{j=i} a_{ij}\varepsilon_{j,n} \quad \ldots\ldots (9)$$

where

$$c_{ii} = \sum_{j=i+1}^{j=N} q_{ij} f'_{ij}(S_{i,n})\,\Delta t / V_{pi} \qquad a_{ii} = 1 - c_{ii} \quad \ldots\ldots (10a)$$

$$c_{ij} = q_{ij} f'_{ij}(S_{j,n})\,\Delta t / V_{pi} \qquad a_{ij} = c_{ij} \quad j < i \quad ..(10b)$$

All c_{ij} are ≥ 0; $a_{ij} \geq 0$ for $j < i$ and $a_{ij} = 0$ for $j > i$. The coefficient c_{ii} is the CFL$_i$ number for block i. The term CFL is $Max(i)\,c_{ii}$. In matrix form, Eq. 9 is

$$\boldsymbol{\varepsilon}_{n+1} = A\boldsymbol{\varepsilon}_n \quad \ldots\ldots (11)$$

which gives

$$\boldsymbol{\varepsilon}_n = A^n\boldsymbol{\varepsilon}_0 \quad \ldots\ldots (12)$$

$$\|\boldsymbol{\varepsilon}_{n+1}\| = \|A\boldsymbol{\varepsilon}_n\| < \|A\|\,\|\boldsymbol{\varepsilon}_n\| \quad \ldots\ldots (13)$$

where the double brackets denote a norm, e.g. Euclidean or maximum.

The stability limit on CFL for Eq. 2 or 9 depends upon the definition of stability. The literature gives different definitions. A common definition in the mathematics literature is that $\boldsymbol{\varepsilon}_n \to 0$ for sufficiently large n. This is satisfied if and only if the spectral radius (maximum absolute eigenvalue) $\rho(A) < 1$.[15] Since the eigenvalues of the triangular A are equal to its diagonal entries $1 - c_{ii}$ and $c_{ii} > 0$, this gives the stability condition

$$\text{CFL} < 2 \quad \ldots\ldots (14)$$

For Eq. 2, with constant $D_i = D$ and $C_i = C$, Hildebrand[13] also gives the stability condition CFL < 2 for the hyperbolic case $D_i = 0$ (his general condition is $2D + C + 2\sqrt{D(D+C)} < 2$).

Todd *et al* required non-oscillatory stability or positive eigenvalues $1 - c_{ii}$. This gives the more restrictive stability condition $c_{ii} < 1$ or

$$\text{CFL} < 1 \quad \ldots\ldots (15)$$

They indicated an effect of whether the flow rates q_x, q_y, q_z in their 3D Cartesian grid were everywhere positive. This requirement CFL < 1 is unaffected by flow directions. The von Neumann method of stability analysis,[13,16] applied to Eq. 2, gives $2D + C < 1$ or CFL = C < 1 for the hyperbolic case. Application of von Neumann's method to Eq. 9 gives CFL < 1 (see the Appendix).

Russell[8] requires the maximum norm ratio $\|\boldsymbol{\varepsilon}_{n+1}\| / \|\boldsymbol{\varepsilon}_n\| < 1$ or $\|A\| < 1$. This gives[8,15]

$$\|A\| = Max(i) \sum_{j=1}^{j=N} |a_{ij}| < 1 \quad \text{(16a)}$$

or

$$\sum_{j<i} c_{ij} + |1 - c_{ii}| < 1 \quad i = 1, 2, .., N \quad \text{(16b)}$$

If all the f'_{ij} values for a given i are assumed equal, then $\sum_{j<i} c_{ij} = c_{ii}$ and Condition 16b gives CFL < 1. The Watts and Rame inflow, outflow, and composite stability conditions, written in terms of Eq. 9 coefficients, are

$$\Theta_{inflow} = \sum_{j<i} c_{ij} < 1 \quad \text{(17a)}$$

$$\Theta_{outflow} = |1 - c_{ii}| < 1 \quad \text{(17b)}$$

$$\Theta_{composite} = \Theta_{inflow} + \Theta_{outflow} < 1 \quad \text{(17c)}$$

These three conditions appear equivalent to Russell's Condition 16b.

In summary, for cocurrent hyperbolic flow, both CFL < 1 and CFL < 2 ensure stability in the sense that $\boldsymbol{\varepsilon}_n \to 0$ for sufficiently large n. The condition CFL < 1 gives positive eigenvalues < 1 and the non-oscillatory stability of Todd *et al*. For 1 < CFL < 2, all eigenvalues are < 1 in absolute value but the dominant eigenvalue can be negative, giving oscillatory stability.

4. Numerical Results for Constant CFL < 1 and CFL < 2

For the 1D miscible flow case with equal grid spacing, $f = S$ and $f' = 1.0$, V_{pi} is constant, $c_{ij} = c_{ii} = c = CFL = q\Delta t / V_{pi}$, and the error Eq. 9 is

$$\varepsilon_{i,n+1} = c\,\varepsilon_{i-1,n} + (1-c)\,\varepsilon_{i,n} \quad \text{(18)}$$

The Appendix gives a short Fortran program which solves Eq. 18. For N=20, c=1.5, and $\varepsilon_{i,0} = .0001\,\delta_{i1}$, the maximum norm $\|\boldsymbol{\varepsilon}_n\|$ increases to a maximum of 15000 at n = 37 and decreases thereafter with all $\varepsilon_{i,n}$ oscillatory. For any N and CFL = c < 2, $\|\boldsymbol{\varepsilon}_n\| \to 0$ for sufficiently large n but it reaches intermediate values increasing astronomically as the CFL approaches 2. For CFL > 2, all $\varepsilon_{i,n}$ increase with n without bound. For CFL < 1, $\|\boldsymbol{\varepsilon}_n\|$ decreases monotonically with n for an arbitrary initial error vector. Russell[8] demonstrates this error growth when CFL > 1. Clearly, when the block CFL numbers are constant, CFL < 1 is necessary for the stability our problems require.

5. The Variable CFL Case

Eq. 9 applies to a 1D Buckley-Leverett (BL) problem where water is injected into an oil reservoir of initial water saturation S_{wc}. Young and Russell[10] reported 1D and 2D BL example problems which exhibited stability for CFL < 2 and instability for CFL > 2. We reproduced their observations and ran a number of other BL problems, generally finding the same result - stability for CFL < 2, for moderate N of about 20 or less.

The question is: why does stability require CFL < 1 in the miscible flow case and allow CFL < 2 in the two-phase case? An obvious major difference between these cases is that the block CFL numbers are constant in the miscible case and variable in the two-phase case. We are not aware of any theoretical analysis of the effect of variable coefficients on the solution of Eq. 9. So we examine this effect by solving Eq. 9 using a spatial variation of CFL$_i$ which is identical to that of the two-phase case.

For saturations behind the front, the BL solution gives $x(S) = utf'(S)/\varphi$, and $f'(S)$ therefore increases linearly with x behind the front. The 1D block CFL number $c_i = q f'_i \Delta t / V_{pi}$ therefore increases linearly with distance behind the front. After breakthrough, c_i is a maximum at i=N and equals $(i/N)c_N$ for i<N. This spatial variation of c_i is invariant with time. As previously mentioned, the maximum c_i value, c_N, is referred to simply as the CFL number.

For this 1D variable CFL case with equal grid spacing, Eq. 9 is

$$\varepsilon_{i,n+1} = c_{i-1}\varepsilon_{i-1,n} + (1-c_i)\varepsilon_{i,n} \quad \text{(19)}$$

Eq. 19 is somewhat more rigorous than the form

$$\varepsilon_{i,n+1} = c_i\varepsilon_{i-1,n} + (1-c_i)\varepsilon_{i,n} \quad \text{(20)}$$

which corresponds to the common assumption of taking a single f' value for flow into or out of a block. However, the difference in these equations is of no consequence in the present discussion regarding the effect of variable c_i. The Appendix Fortran program solves Eq. 19. Runs were made for N = 20 and $\varepsilon_{i,0} = .0001\,\delta_{i1}$ or $\|\boldsymbol{\varepsilon}_0\| = .0001$. The maximum error, $\delta = Max(n)\,|\varepsilon_{i,n}|$, occurs in block N. For runs with CFL < 1.99, δ is essentially 0 (.000055 for CFL= 1.99). This compares with δ= 15000 for CFL = 1.5 for the constant CFL case discussed above. For a CFL = 2.01, $\varepsilon_{N,n}$ oscillates unstably with increasing, unbounded amplitude. For a given, small delta, CFL must be decreased from 2 as N is increased above 20.

This stability of the 1D BL case for CFL < 2 and moderate N is simply a consequence of the variable block CFL numbers, which decline linearly with distance upstream from the producer. This stabilizing effect is greater in 2D or 3D BL cases since both rate q and f' decrease upstream from the producer. The fact remains that instability (oscillations) may occur for CFL > 1 in regions where the block CFL numbers are uniform, or nearly so. Such regions can arise in countercurrent flow and other flow situations.

There appears to be no relation between this BL problem stability for CFL < 2 and the Conditions 16 or 17. They predict a stable CFL limit approaching 2 only when the inflow term $\sum_{j<i} c_{ij} \ll c_{ii}$. But in the BL case, the controlling CFL number occurs in the region where $\sum_{j<i} c_{ij}$ is very nearly equal to c_{ii}.

6. Conclusions

The saturation error equation for stability analysis is derived for miscible or two-phase cocurrent hyperbolic flow in a general grid. For miscible or two-phase flow with spatially uniform block CFL numbers, the Todd *et al* conclusions apply:

- CFL < 1 is required for non-oscillatory stability.
- 1 < CFL < 2 gives oscillatory stability.
- Our reservoir problems require non-oscillatory stability.

For the 1D Buckley Leverett problem, the time-invariant spatial variation of block CFL number results in stability for moderate N for CFL < 2. This stabilizing effect of variable block CFL number is greater in 2D and 3D BL type displacements.

Nomenclature

A = cross-sectional area normal to 1D flow
a_{ij} = elements of the error amplification matrix A (Eq. 11)
c_{ij} = see Eq. 10
CFL = Courant-Friedrichs-Lewy number
C = convective coefficient in Eq. 2
D = diffusive coefficient in Eq. 2
F = component of generalized CFL number (Eq. 1)
f = fractional flow
f_{ij} = fractional flow for flow between blocks i and j
f' = df/dS where S is displacing phase saturation
$\hat{i}$ = $\sqrt{-1}$
k = absolute permeability
k_r = relative permeability
N = total number of grid blocks
P_{cgo} = gas-oil capillary pressure
P'_{cgo} = dP_{cgo}/dS_g
q = total flow rate, reservoir volume/day
q_i = see Eq. 6
q_{ij} = total flow rate between blocks i and j
S = saturation or mol fraction
t = time
T = transmissibility, $k\Delta y\Delta z/\Delta x$
T_{ij} = transmissibility connecting blocks i and j
u = q/A
V_p = grid block pore volume
x, y, z = Cartesian coordinates
Z = depth, measured vertically downward

Greek

δ_{ij} = Dirac delta function, 1 if $i=j$, 0 if not
δ = $\text{Max}(n)|\varepsilon_{N,n}|$
$\Delta x, \Delta y, \Delta z$ = grid block dimensions
Δt = time step
$\varepsilon_{i,n}$ = error in $S_{i,n}$
$\boldsymbol{\varepsilon}_n$ = vector $\{\varepsilon_{i,n}\}$, $i=1,N$
γ_g = gas phase density gradient
γ_o = oil phase density gradient
λ_g = gas phase mobility, k_{rg}/μ_g
λ_o = oil phase mobility, k_{ro}/μ_o
λ_t = total mobility
φ = porosity, fraction
Ψ = $\lambda_o\lambda_g/(\lambda_o+\lambda_g)$

Subscripts

i = grid block index
j = grid block index
g = gas phase
n = time step number
o = oil phase
w = water phase
x,y,z = Cartesian directions

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Appendix

The following is a brief derivation of the well known diffusion-convection equation for 1D, two-phase (gas-oil), cocurrent flow. Darcy's law gives gas and oil flow rates in the direction of increasing *x* as

$$q_g = -kA\lambda_g(\partial p/\partial x - \gamma_g dZ/dx + \partial P_{cgo}/\partial x) \quad \text{(A-1a)}$$

$$q_o = -kA\lambda_o(\partial p/\partial x - \gamma_o dZ/dx) \quad \text{(A-1b)}$$

Eliminating $\partial p/\partial x$ gives

$$q_g = qf_g - kA\psi\partial P_{cgo}/\partial x \quad \text{(A-2)}$$

where

$$f_g = \frac{\lambda_g}{\lambda_t}\left(1 - \frac{kA(\gamma_o - \gamma_g)\lambda_o dZ/dz}{q}\right) \quad \text{(A-3)}$$

If phase densities are assumed constant and interphase mass transfer is neglected, the continuity equation for gas is

$$-\partial q_g/\partial x = \varphi A\partial S_g/\partial t \quad \text{(A-4)}$$

and substituting q_g from Eq. A-2 gives

$$\frac{\partial}{\partial x}\left(D\frac{\partial S_g}{\partial x}\right) - C\frac{\partial S_g}{\partial x} = \frac{\partial S_g}{\partial t} \quad \text{(A-5)}$$

where D is $kP'_{cgo}\psi/\varphi$ and C is $q f'_g/A\varphi$. The corresponding explicit difference equation is Eq. 2. The discrete form of Eq. A-3 for flows q_{ij} and q_{gij} defined as flows from block *i* to block *j* is

$$f_{gij} = \frac{\lambda_g}{\lambda_t}\left(1 - \frac{T_{ij}\lambda_o}{q_{ij}}(\gamma_o - \gamma_g)(Z_j - Z_i)\right) \quad \text{(A-6)}$$

where mobilities are evaluated at the upstream block saturations.

Section 2 implies that the grid block numbering mentioned is necessary to obtain a triangular matrix *A* (Eq. 9 or 11) with eigenvalues equal to its diagonal elements. However, the eigenvalues of the matrix *A* are its diagonal elements regardless of the ordering of the blocks - i.e. regardless of whether *A* is triangular or not. We can start with a random numbering of the grid blocks and a resulting matrix *A* which is obviously non-triangular. Then a succession of row and corresponding column interchanges leads to the triangular matrix *A* of Eq. 9. Each interchange corresponds to a renumbering of the blocks. But such interchanges do not change the eigenvalues of the matrix,[12] or its diagonal entries. Therefore the original non-triangular matrix and the final triangular matrix have the same eigenvalues and diagonal entries.

The von Neumann method of stability analysis substitutes $\lambda^n e^{\hat{i}\beta_m j}$ for $\varepsilon_{j,n}$ in Eq. 9 and requires that $|\lambda| < 1$ for all real values of β_m. This gives

$$\lambda = \sum_{j=1}^{j=i-1} c_{ij}e^{\hat{i}(j-i)\beta_m} + 1 - c_{ii} \quad \text{(A-7)}$$

Choosing all $(j-i)\beta_m$ as odd multiples of π and even multiples of π gives, respectively,

$$c_{ii} + \sum_{j<i} c_{ij} < 2 \quad \text{(A-8)}$$

$$c_{ii} > \sum_{j<i} c_{ij} \quad \text{. . . . . (A-9)}$$

If all $f_{ij}^{'}$ values in c_{ij} are assumed equal to some $f_i^{'}$, then $c_{ii} = \sum_{j<1} c_{ij}$, Condition A-9 is satisfied, and Condition A-8 reduces to CFL < 1. Conditions A-8 and A-9 taken together give $\sum_{j<1} c_{ij} < 1$ (for any positive *a, b, b* + *a* < 2 and *b* - *a* < 0 require that *b* < 1). Russell's Condition 16b clearly requires $\sum_{j<1} c_{ij} < 1$ and also leads to Condition A-9. The time step Δt cancels out in this Condition A-9, and the block coefficients may violate it. Several responses can be devised for this dilemma in the AIM or Impes case, including relaxing the limits < 1 of Condition 16 and < 2 of Condition A-8.[10,11]

In many 1D two-phase Buckley-Leverett simulations, changes rather than stable step size must be used during the run prior to breakthrough.[10] Only behavior after breakthrough was considered in the discussion of stable CFL number for the BL case. The maximum change in saturation over the (stable) time step can be shown to equal $S_{N-1,n} - S_{N,n}$, which is frequently a small value the order of .001 - .005.

The simulator BL numerical test problems were run as follows. Each time step was completed with no flow reversals and with a time step such that the maximum CFL_i number, $q f_i^{'} \Delta t / V_{pi}$, exactly equalled the value stated in the discussion. Impes transmissibilities were recalculated within the time step when necessary to cope with flow reversals. The time step size changed from step to step. The model calculated k_r and $f^{'}$ using analytical expressions of $\square_r$.

As an addendum to the Section 5 discussion, it is not necessary that *N* be significant or large for errors to build up and cause instability in a block having a CFL number > 1. For example, with *N* = 1 or 2, inject a mixture of water and oil into block 1, using an implicit production block in cell *N*+1. Use the Young and Russell or any other mobility data. After some time, change the injection mix proportion; then after some time change it again. For CFL > 1, the block 1 (and 2, if *N* =2) response is oscillatory; the oscillation amplitude increases with the CFL number and with the strength of the perturbations (changes in injection mix). No oscillations occur if CFL < 1. To the contrary, if *N* is such that perturbations reach a block of CFL > 1 through a path of upstream blocks where their block CFL numbers decline sufficiently in the upstream direction, then the block may respond stably.

Fortran Program for Solution of Eqs. 18-20

```
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION EOLD(1000),ENEW(1000),C(1000)

N=20
IFLAG=2  !  solve Eq. 19
 CFL=1.99
EFEED=0.
DO I=1,N
  EOLD(I)=0.
ENDDO
EOLD(1)=.0001   ! whatever

IF (IFLAG .EQ. 0) THEN
  DO I = 1,N   ! Eq. 18
    C(I)=1.
  ENDDO
ELSE
  C(N)=1.    ! Eqs. 19 or 20
  U1=1./N
  DO I=1,N-1
    C(I)=U1*I
  ENDDO
ENDIF

DO ITIME=1,20*N
  UC=CFL*C(1)
  ENEW(1)=UC*EFEED+(1.-UC)*EOLD(1)
  IF (IFLAG .LE. 1) THEN
    DO I=2,N    ! Eqs. 18 or 20
      UC= CFL*C(I)
      ENEW(I)=UC*EOLD(I-1)+(1.-UC)*EOLD(I)
    ENDDO
  ELSE
    DO I=2,N    ! Eq. 19

ENEW(I)=CFL*C(I-1)*EOLD(I-1)+(1.-CFL*C(I))*EOLD(I)
    ENDDO
  ENDIF
  UNORM=0.
  DO I=1,N
    UNORM=MAX(UNORM,ABS(ENEW(I)))
    EOLD(I)=ENEW(I)
  ENDDO        ! print ..
ENDDO
END
```

SI Metric Conversion Factors

bbl	x 1.589 873	E-01	= m^3
ft^3	x 2.831 685	E-02	= m^3
lbm	x 4.535 924	E-01	= kg
psi	x 6.894 757	E+00	= kPa