Determination of Aquifer Influence Functions From Field Data

K. H. COATS
L. A. RAPOPORT
MEMBERS AIME
J. R. McCORD
W. P. DREWS

JERSEY PRODUCTION RESEARCH CO.
TULSA, OKLA.

ESSO RESEARCH & ENGINEERING CO.
MADISON, N. J.

ABSTRACT

Water movement about an oil or gas reservoir can be predicted provided an aquifer influence function is known. This function is generally determined from an electric analyzer study or by fitting an idealized mathematical model to field data. A problem arises in determining that influence function which best reproduces field data and satisfies certain smoothness requirements, without making any idealizations concerning aquifer geometry and heterogeneity. This study shows the problem to be solvable through the linear programming technique and presents applications to one oil reservoir and two gas storage reservoirs. In addition, a theorem is stated and proven which confirms influence function smoothness requirements heretofore accepted on physical intuition and establishes additional smoothness constraints formerly not recognized.

INTRODUCTION

The pressure decline accompanying production results in water encroachment into oil and gas reservoirs situated adjacent to aquifers. The importance of this water movement derives from the significant dependence of production rate upon reservoir pressure and of pressure, in turn, upon water encroachment.

Pressure and rate of water movement are uniquely related through an "influence" function which may be in the form of reservoir pressure response to a unit water encroachment rate or of water movement caused by a unit reservoir pressure drop. This influence function reflects the heterogeneity and geometry of the aquifer and is therefore particular to each reservoir. If the influence function is known, then reservoir pressure may be predicted from a production rate schedule, or vice-versa, with the aid of the material balance equation which relates water influx to reservoir production and pressure.

The problem of determining this influence function for a given reservoir has been approached in three different ways. Van Everdingen and Hurst calculated and tabulated certain influence functions pertinent to idealized mathematical models which satisfy simplifying assumptions such as aquifer homogeneity and elementary reservoir-aquifer geometries.[1] Other studies developed additional tables for a variety of elementary reservoir-aquifer geometrical configurations and demonstrated their validity and utility.[3,4] Hicks, *et al.* described the use of an electrical network to determine an influence function giving "best" match of field history and allowing prediction of future performance.[2] Finally, Hutchinson and Sikora[5] and Katz, Tek and Jones[6] attempted to develop a calculation method for deriving the influence function directly from field data with no idealizing assumptions concerning aquifer geometry and homogeneity. They encountered difficulties with inaccuracies in field data and with satisfaction of well-known smoothness requirements of the influence function.

This study is concerned with the determination of influence functions from field data and, more specifically, treats the following problem: given field pressure-production data of arbitrary accuracy, how does one determine the influence function which, subject to satisfaction of known smoothness requirements, gives best agreement with field history? A rigorous solution to this problem is obtained through linear programming as described below. Example applications to one oil and two gas storage reservoirs are presented. In addition, a theorem is proved which confirms influence function smoothness requirements heretofore accepted on physical intuition and establishes additional smoothness requirements formerly not recognized.

DEFINITION AND PROPERTIES OF THE INFLUENCE FUNCTION

If the influence function $F(t)$ denotes the reservoir pressure response to a unit rate of water influx, then the reservoir pressure response to a varying rate of water movement, $e_w(t)$, is given by[1,3,4]

$$p_o - p(t) = \int_0^t e_w(t-\tau) \frac{dF(\tau)}{d\tau} d\tau \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The integral may be approximated by a summation to give

$$p_o - p_i = \sum_{j=1}^{j=i} (e_{w,i-j+1} - e_{w,i-j}) F_j \quad . \quad . \quad . \quad . \quad (2)$$

or, equivalently,

$$p_o - p_i = \sum_{j=1}^{i} e_{w,i-j+1} \Delta F_j \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\Delta F_j = F_j - F_{j-1}$. In these equations, e_{wo} and F_o are both zero.

We will now consider the situation where $F(t)$ is unknown but $p(t)$ and $e_w(t)$ are known from field data.

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[1]References given at end of paper.

Actually, the influx $e_w(t)$ is calculated from pressure-production data and a volumetric balance equation. The determination of F_i appears simple since Eq. 2 may be written successively for $i=1, 2, \ldots n$ and solved for $F_1, F_2, \ldots F_n$, respectively. For example, for $i=1, 2, 3$, Eq. 2 gives

$$p_o - p_1 = e_{w1}F_1$$
$$p_o - p_2 = (e_{w2} - e_{w1})F_1 + e_{w1}F_2$$
$$p_o - p_3 = (e_{w3} - e_{w2})F_1 + (e_{w2} - e_{w1})F_2 + e_{w1}F_3, \quad \ldots \quad (4)$$

from which F_1, F_2, and F_3 can easily be calculated. However, since the pressure p and influx e_w data may be in error, we have no guarantee that the calculated F values will satisfy certain well-known smoothness contraints. These constraints have generally been stated as[5,6]

$$F(t)>0$$
$$dF(t)/dt \geq 0$$
$$d^2F(t)/dt^2 \leq 0 \quad \ldots \quad (5)$$

holding true for all time greater than zero. Their physical interpretation is that a constant rate of water injection into a porous medium of arbitrary geometry and heterogeneity will result in an injection "face" pressure change which is always increasing but at a steadily decreasing rate. Darcy flow and an initial state of equilibrium in the porous medium are assumed. These conditions are indeed evidenced by all analytical solutions $F(t)$ for homogeneous aquifers of elementary geometries. While physical intuition extends their validity to aquifers of arbitrary geometry and heterogeneity, no proof of their necessity iin this general case has been given.

We now state a theorem which is given in more mathematical terms and proven in the Appendix; the theorem presumes single-phase, Darcy flow of a slightly compressible fluid:

If an aquifer of arbitrary geometry and heterogeneity, initially at equilibrium, supplies fluid at a constant rate e_w through a surface (i.e., reservoir-aquifer interface) then the average pressure change, $F(t) = p_o - p(t)$, over that surface satisfies the conditions*

$$e_wF(t)>0$$
$$e_wF^{2k-1}(t) \geq 0 \qquad k = 1,2,3, \quad \ldots \quad (6)$$
$$e_wF^{2k}(t) \leq$$

In addition, $F(t)$ and all its derivatives are continuous functions of time, t.

Thus the conditions Eq. 5 are necessary but not sufficient; the conditions of Eq. 6 state that in fact *all* odd derivatives are non-negative and *all* even derivatives are non-positive for positive e_w. If e_w is negative, signs are simply reversed.

Experience with the direct calculation of Eq. 4 shows it to be unstable in the sense that errors in field data result in some negative F_i and in progessively greater oscillations in F_i at increasing values of i (time).[5,6] The irregular or oscillating $F(t)$ function obtained in this manner is useless for prediction purposes since it forms no basis for the extrapolation to larger time which is necessary for prediction.

LINEAR PROGRAMMING FORMULATION

The problem of determining $F(t)$ may now be stated as follows: field data p_i and e_{wi} are given at $n + 1$ equally spaced points in time with p_o corresponding to an initial state of aquifer equilibrium. An influence function F_i, $i = 1,2, \ldots, n$, is desired which satisfies constraints (Eq. 6) and minimizes the sum of deviations.

$$\sum_{i=1}^{n} \left| p_o - p_i - \sum_{j=1}^{i} e_{w\,i-j+1} \Delta F_j \right| \quad \ldots \quad (7)$$

We can restate the above as an ordinary linear programming problem. We allow slack in Eq. 2,

$$\sum_{j=1}^{i} \alpha_{i,j}F_j + u_i - v_i = b_i,\ i = 1,2, \ldots, n, \quad \ldots \quad (8)$$

impose the constraints

$$F_i \geq 0 \quad i = 1,2, \ldots, n, \quad \ldots \quad (9)$$

$$\sum_{j=n-i}^{n} c_{i,j}F_j \geq 0, i = 1,2, \ldots, n, \quad \ldots \quad (10)$$

and demand minimization of the objective function

$$\sum_{i=1}^{n} (u_i + v_i). \quad \ldots \quad (11)$$

Definitions $\alpha_{i,j} = e_{w\,i-j+1} - e_{w\,i-j}$ and $b_i = p_o - p_i$ are made to simplify notation. The $c_{i,j}$ are the coefficients in the ith order difference form representing the ith order derivative in Eq. 6. Thus, for $i=1$, Eq. 10 is

$$F_n - F_{n-1} \geq 0$$

and for $i = 2$,

$$-F_n + 2F_{n-1} + F_{n-2} \geq 0.$$

The u_i and v_i are "slack" variables whose difference represents the deviation between calculated pressure change $\Sigma\alpha_{i,j}F_j$ and observed pressure change b_i. This deviation is represented by a difference of two variables because the deviation may be positive or negative while the linear programming method handles only positive variables. The n constraints (Eq. 10) are equivalent to the $n(n+1)/2$ constraints (Eq. 6) constituted by n first order difference constraints, $n - 1$ second order differences, ..., and 1 nth order difference $(1 + 2 + 3 + \ldots + n = [n(n + 1)]/2)$. Note that the order of the highest derivative expressible in difference form with $n + 1$ F_i values is n.

The problem defined by Eqs. 8 through 11 is solvable by the linear programming technique for which computer codes are available on practically all digital computers.[13] These programs are available from the computer manufacturers or through computing consulting firms. Required input data are e_{wi} and b_i for $i=1, 2, \ldots, n$. Values for c_{ij} are also needed, although practical considerations simplify their selection considerably as discussed below. Minimization of the objective function (Eq. 11) is equivalent to minimization of Eq. 7; in the final results of the computation either u_i or v_i will be zero for each value of i.

A computational difficulty arises in connection with the constraints (Eq. 10) in that the coefficients $c_{i,j}$ cover a very large range as the order i of the difference form grows large. The coefficients range from (disregarding sign) unity to $(i!)/(\frac{i}{2}!)^2$ for even i, or $(i!)/(\frac{i-1}{2}!)^2 (\frac{i+1}{2})$ for odd i. Thus, the nth (Eq. 10) will have coefficients ranging from 1 to about 13,000, 184,000 and 1.38×10^9 for $n=16$, 20 and 40, respectively. This suggests that differences of larger order than about 15 may give round-off error difficulty in the linear programming solution of Eqs. 8 through 11 with a machine carrying eight digits. Fortu-

*e_w appears here simply for the purpose of sign definition.

nately, our experience to date indicates that, while the inclusion of third order difference constraints is generally desirable, the higher order difference constraints may be ignored.

If Eq. 3 rather than 2 is employed and only difference constraints up to third order are imposed, then Eqs. 8 through 11 become

$$\sum_{j=1}^{i} e_{w(i-j+1)} X_j + u_i - v_i = b_i \qquad i = 1,2,\ldots,n, \quad (12)$$

$$X_i \geq 0 \quad i = 1,2,\ldots,n. \quad (13)$$

$$X_{n-1} - X_n \geq 0, \quad (14)$$

$$X_{i+1} - 2X_i + X_{i-1} \geq 0 \quad i=2,3,\ldots,n-1,$$

$$\sum_{i=1}^{n} (u_i + v_i) = \text{minimum} \quad (15)$$

If only the first and second order derivative constraints on $F(t)$ are imposed then Eq. 14 is replaced by

$$X_{i-1} - X_i \geq 0 \quad i = 2,3,\ldots,n \quad (16)$$

Here X_i denotes $\Delta F_i = F_i - F_{i-1}$; F_i equals $\sum_{j=1}^{i} X_j$.

The generalization of Eqs. 12 through 15 to include all constraints of order $k(\leq n)$ and lower involves only $n-1$ constraints regardless of the value of k. Each difference of order less than k requires only one constraint, specifically that difference expressed at as large a time index i as possible considering the span of its members. Thus the first difference required is $X_n - X_{n-1}$, the second difference is $X_n - 2X_{n-1} + X_{n-2}$, the required third difference is $X_n - 3X_{n-1} + 3X_{n-2} - X_{n-3}$, etc. In addition to one constraint for each difference of order $1,2,\ldots,k-1$, all possible differences (considering the span) of order k are required, except the one which requires the undefined X_0. Since a difference of order k requires values at $k+1$ points, $n-k+1$ constraints of order k are expressible with $n+1$ points and the total number of constraints is $(k-1) + (n-k)$ or $n-1$. These $n-1$ constraints guarantee satisfaction of each difference equivalent of Eq. 6 of order k and less at all time points for which the difference is expressible. For example, Eq. 14 ensures that $X_{n-2} - X_{n-1} \geq 0$, $X_{n-3} - X_{n-2} \geq 0$, etc.

Application of the linear programming method to Eqs. 8 through 11 or 12 through 15 yields F_i values for $i=1,2,\ldots,n$. Prediction of field performance for $m\ (>n)$ time increments requires extrapolation of the F_i values into the range $n<i\leq m$. This extrapolation is discussed in the literature.[4-6] The $F(t)$ function will reach a constant value at large time for an outcropping aquifer; a closed aquifer of any shape will yield an $F(t)$ function increasing linearly with time at large time. At large time, $F(t)$ will approach the form $a_1 \log_e(t) + b_1$ for an infinite radial aquifer and $a_2\sqrt{t} + b_2$ for an infinite linear aquifer, where a_1, a_2, b_1, b_2 are constants. Thus, if the last few F_i values exhibit any of the above characteristic variations with time, then the extrapolation is straightforward.* Otherwise, predictions may be carried out for two or more extrapolations lying between the extreme or limiting extrapolations corresponding to a finite, closed aquifer and an outcropping aquifer.

APPLICATIONS

The method just described has been applied to two gas-storage reservoirs and one oil reservoir. Data for gas Fields A and C were obtained from Ref. 4. Field A is a gas reservoir which after depletion was shut in for several years prior to conversion to gas storage. Pressure and gas in place were 408 psig (wellhead) and 1,000,000 Mcf at the end of the shut-in period. These conditions were used as the initial or zero time point for determination of the influence function. Observed reservoir pressure is plotted in Fig. 1 and gas-occupied pore volume is given in Fig. 2. The pore volume was calculated from pressure and gas in place data as

$$V_i = \frac{z_i n_i RT}{p_i}, \quad (17)$$

where $z\ (=0.9975 - 1.25\times10^{-4}\,p)$ is the gas compressibility factor and n is gas in place. The tenfold change in pore volume emphasizes the necessity of accounting for water movement in this case.

The influx terms e_{wi} were calculated as

$$e_{wi} = V_{i-1} - V_i, \quad (18)$$

for the 20 monthly time increments, $i=1,2,\ldots,20$. These data along with pressure data $p_0 - p_i$ were fed into an IBM 7094 computer and Eqs. 12 through 15 were solved by the LP-90 program. This process was then repeated with the third-order difference constraints (Eq. 14) replaced by the second-order constraints (Eq. 16). Each calculation required less than two minutes of 7094 time. The influence functions determined for these two cases are given in Fig. 3. Also shown in Fig. 3 is the influence function determined by Katz, *et al.* for this same field.[4]

The influence function given by the second-order constraints was simply four straight-line segments; thus Fig. 3 shows only the slope breakpoints for this curve. The two influence functions are very close, the only significant difference being their extrapolated slopes. The solid curve in Fig. 3 appears somewhat irregular in spite of the fact

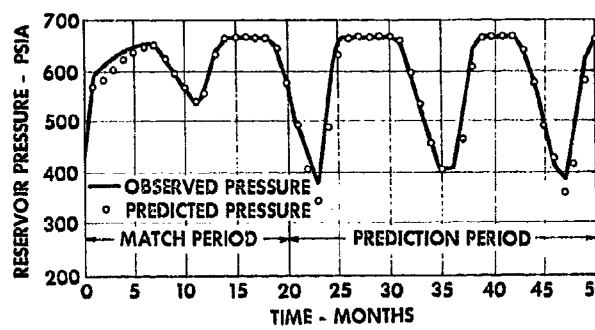

FIG. 1—FIELD A OBSERVED AND PREDICTED PRESSURE.

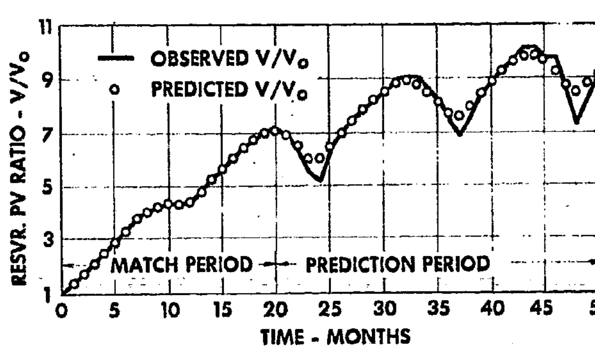

FIG. 2—FIELD A OBSERVED AND PREDICTED PORE VOLUME.

*This statement must be qualified by the observation discussed later that erroneous field data as well as actual aquifer characteristics may result in one of the characteristic types of variation in terminal F_i values.

that the smoothness constraint (Eq. 6) on its first three derivatives are satisfied. A linear programming solution was therefore obtained with fourth-order difference constraints imposed. The resulting change in the influence function is not discernible with the scale of Fig. 3.

Combination of Eqs. 17 and 18 gives e_{wt} in terms of pressure and gas in place. Substitution of that result into Eq. 3 then gives a single equation relating pressure, gas in place and influence function values.[4,6,7] This equation is a quadratic in pressure from which pressure can be calculated if gas in place and influence function values are given. The influence function corresponding to third-order difference constraints was employed, extrapolated in the straight-line manner shown in Fig. 3. Figs. 1 and 2 compare observed reservoir pressure and pore volume with predicted values calculated in this manner. Katz *et al.* found in their analysis of Field A that a mathematical model corresponding to a finite aquifer of exterior radius 14 times reservoir radius gave best fit of performance.[4] Use of their influence function method similarly gave a curve having a terminal linear segment indicating a finite aquifer. These facts augmented by the terminal linear segment of our influence function and the excellent match of field performance obtained by the linear extrapolation indicate a finite aquifer. The average 2.46 per cent deviation between our predicted and observed pressures compares to an average 3 per cent deviation obtained by Katz *et al.*[4]

Field C is a Michigan gas reservoir which was produced for six years after discovery prior to conversion to gas-storage operation. Field description and detailed data are given in Ref. 4. Figs. 4 and 5 show observed reservoir pressure and pore volume (Eq. 17) as functions of time; zero time corresponds to discovery date. Data for the first 36 two-month time increments were employed along with third-order difference constraints (Eqs. 12 through 15) to determine the influence function shown in Fig. 6. The last few values of F_t do not vary as ln t, t or $\sqrt{t}$. Thus, predictions of reservoir pressure from gas-in-place data were

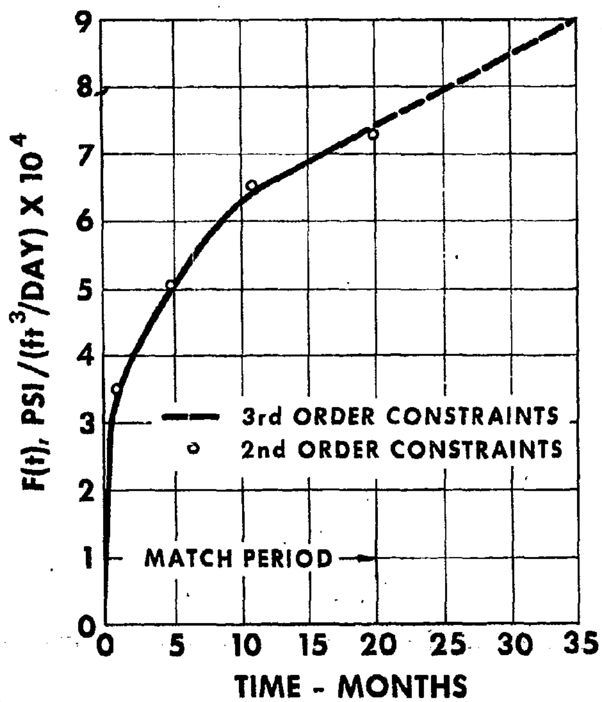

FIG. 3—FIELD A INFLUENCE FUNCTIONS.

carried out for the two limiting extrapolations shown in Fig. 6, one corresponding to a closed aquifer ($F=at+b$), the other to an outcropping aquifer (F=constant). The observed reservoir pressure and pore volume are compared to the predicted values for a closed aquifer extrapolation in Figs. 4 and 5. The average pressure deviation over all points of Fig. 4 is 3 per cent, with an average of 5 per cent in the prediction period alone. The erroneously large variations in the predicted pore volume curve

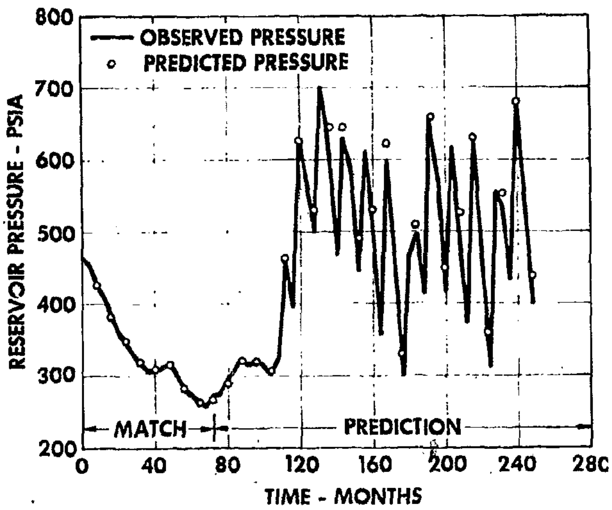

FIG. 4—FIELD C OBSERVED AND PREDICTED PRESSURE.

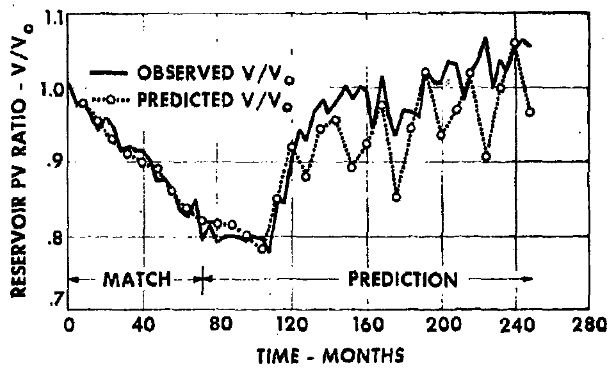

FIG. 5—FIELD C OBSERVED AND PREDICTED PORE VOLUME.

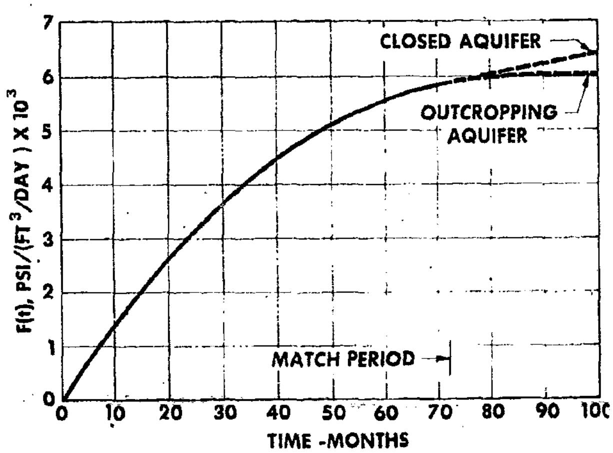

FIG. 6—FIELD C INFLUENCE FUNCTION.

of Fig. 5 indicate that the influence function corresponds to a finite aquifer which is too large. Use of the influence function corresponding to an outcropping aquifer gave an average of 4.5 per cent deviation between predicted and observed pressures over all points.

Field B is the Torchlight Tensleep oil reservoir for which Stewart *et al.* compared several methods of calculating water influx.[8] The pressure-production data shown in Fig. 7 were obtained from their paper. Water influx data e_{wi} were obtained from that data and the material balance equation. Three influence functions were obtained through solution of Eqs. 12 through 15. The first two utilized the first 24 months' data; the first was calculated with second-order constraints (Eq. 16) imposed, the second with third-order constraints (Eq. 14) employed. The third $F(t)$ was calculated from Eqs. 12 through 15 (third-order constraints) for the first 40 months of data. The three influence functions are plotted in Fig. 9.

The influence function corresponding to second-order constraints was again composed of straight-line segments so that only the slope break-points are plotted in Fig. 9. The last few values of F_i were constant, indicating an outcropping aquifer. The 24-month influence function satisfying third-order constraints is shown by the solid line of Fig. 9. The last 10 F_i values fell on a straight line, thus indicating a finite aquifer. The 40-month influence function, shown by the dashed line, agreed closely with the 24-month $F(t)$ over the first 24 months. However, the 40-month $F(t)$ deviated considerably from the extrapolated 24-month $F(t)$ during the period from 24 to 40 months. The last few F_i values of the 40-month influence function again fell on a straight line; this influence function contradicts the implication of the 24-month $F(t)$ that the finiteness of the aquifer corresponds to a quasi-steady-state response [linear $F(t)$ of non-zero slope] to a constant e_w after only 14 months. This contradictory terminal behavior of the two influence functions indicates that erroneous field data as well as actual aquifer characteristics can result in characteristic types of terminal $F(t)$ variation.

Predictions of Torchlight performance were performed using the 24-month (third-order constraints) and 40-month influence functions, each extrapolated as straight lines with slopes of their terminal linear segments. The material balance equation as given by Stewart, *et al.* was employed along with Eq. 3. Their initial oil-in-place and compressibility data of 6.1 million STB and 12.4×10^{-6} 1/psi, respectively, were employed. "Given" data consisted of oil production rate for the first 27 months (through Jan., 1950). After that date production rate was maintained at 500 STB/D, provided field pressure was sufficient to maintain it with a field-averaged PI of 3.2 and minimum well pressure of 15 psia.[8] If pressure was not sufficient then pressure and production rate were related by the PI factor, i.e., rate $= 3.2\ (p - 15)$. Thus both production rate and pressure were predicted for time greater than 27 months.

Fig. 7 shows rate and pressure predicted from the extrapolated 24-month $F(t)$. The predicted pressures and rates decrease significantly more rapidly than was observed. The prediction based on the 40-month $F(t)$, however, is a virtually identical match of field performance as shown in Fig. 8.

Stewart, *et al.* obtained the best predictions of field performance from the van Everdingen, *et al.*[12] and electric analyzer methods. The former method was applied utilizing data over all 67 months in order to fix parameters in the mathematical model. The resulting prediction was comparable to that shown in Fig. 8. Stewart, *et al.* state that "although a reasonably accurate forecast from Nov., 1949, (month 25) was made using cumulative data through May, 1953, (month 67) to fix the values of (model parameters and initial oil in place), a reasonable forecast would have been unlikely based only on data available to Nov., 1949". Two analyzer studies were performed, a Jan., 1950, study utilizing data over the 24-month period and an Aug., 1953, study utilizing all 67 months' data. The first analyzer study yielded predicted rates during the 50 to 67 month period more than 30 per cent lower than those predicted here from the 24-month influence function (Fig. 7). The later analyzer study matched performance in a manner comparable to the prediction obtained from the 40-month influence function (Fig. 8).

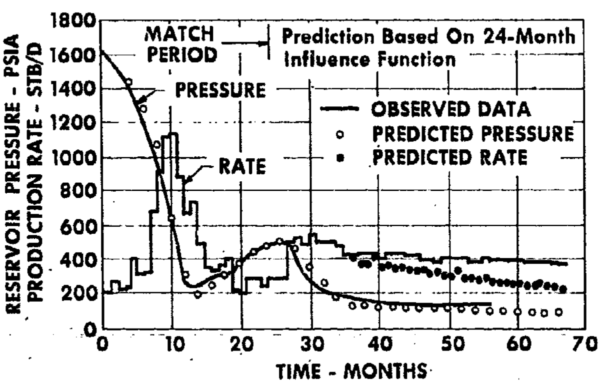

Fig. 7—Field C Actual and Predicted Performance.

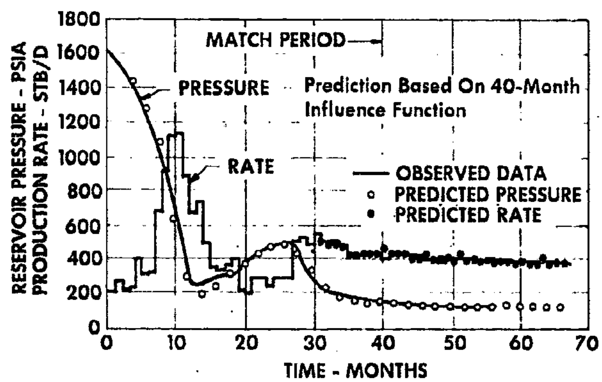

Fig. 8—Field B Actual and Predicted Performance.

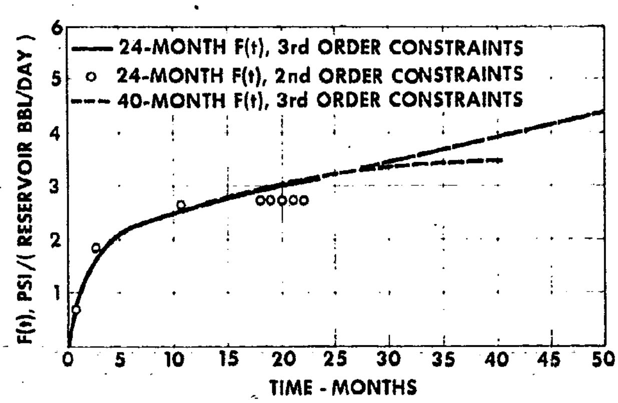

Fig. 9—Field B Influence Functions.

DISCUSSION OF RESULTS

Figs. 3 and 9 show that use of the heretofore proposed second-order constraints (Eq. 16) resulted in influence functions composed of several straight-line segments, i.e., linear segments of three or more points. The curves violate the theorem stated above in that the first derivative $dP(t)/dt$ is discontinuous and the third derivative (difference) oscillates in sign across the break-points. The imposition of third-order constraints was necessary to eliminate these linear segments. These third-order constraints will prevent more than one such segment but allow a "terminal" linear segment of any number of points. In fact, a straight line, $F_i = ai$ where a is constant, satisfies *all* constraints (Eq. 6) and is a possible solution of Eqs. 8 through 11 or 12 through 15.

The imposition of difference constraints only up to the kth order where $k<n$, results in the probability that difference constraints (Eq. 6) of order i, where $k<i\leq n$, will not be satisfied. The pertinent question then is whether use of $k+1$ order constraints would result in significantly different influence function and subsequent prediction than those obtained from k-order constraints. Our experience to date indicates that third-order constraints are necessary to avoid the linear segments generally found in influence functions obtained from second-order constraints but that fourth and higher-order constraints negligibly alter the curve obtained using third-order constraints.

In the above applications the sum of the deviations between observed and calculated pressures at all points $i=1,2,\ldots,n$ of the match period was minimized (Eqs. 8 and 11 or 12 and 15). If certain data points are of questionable accuracy their effects on the derived influence function can be eliminated by simply removing from Eq. 8 or 12, the equations corresponding to them. This is, in a sense, smoothing of data. Alternatively, all data may be smoothed by, say, drawing reasonably smooth curves through time plots of gas-field pore volume or oil-field water influx. The former curves are more likely to exhibit erroneous jumps due to the direct dependence of pore volume upon the two measured variables, gas in place and reservoir pressure. The sensitivity of the derived influence function to data smoothing or to minimization of calculated pressure deviations at only selected points should be investigated.

The essential point of this study is that the given method for deriving $F(t)$ does not require any smoothing of data. While smoothing may be desirable, it is not necessary for workability of the method. The linear programming solution will give that influence function which rigorously satisfies constraints and best matches the data, however good or bad those data are.

While a comparison of the three approaches to calculation of water influx was not a purpose of this study, some comments on this matter are in order. Performance predictions for a newly discovered reservoir or for a planned aquifer storage reservoir[1] obviously require the mathematical model (e.g., van Everdingen and Hurst) approach since field data are necessary for application of the reservoir (electric) analyzer or influence function (this study) approaches.

We feel that, when field data are available, the influence function approach of this study is superior to the mathematical model method, since it is simpler to apply and is technically preferable in that no idealizations concerning geometry and heterogeneity are required. Use of a mathematical model requires choice of a geometry, aquifer extent, boundary conditions, and finally selection of "optimum" values of the (at least) two parameters or groups peculiar to the model. The influence function approach requires only a choice of method of extrapolation.

The disadvantages of the analyzer relative to the influence function method are the original investment of time and money in equipment, the time and skill necessary for trial and error determination of resistor settings and the uncertainty, when finished as to whether the best possible match of field history was obtained. Relative advantages of the analyzer are the ease of extrapolation of the influence function corresponding to the resistor settings and the ability of the analyzer to simulate flow and production by well in the reservoir as well as the aquifer. For gas or gas storage reservoirs and some oil reservoirs where a single average field pressure is of prime concern, the influence function method is probably superior, while the analyzer would be preferable for large oil reservoirs where pressure distribution around and through the reservoir itself is desired.

The linear programming approach given here for derivation of the self-influence function (effect of a field's production on its own pressure) is immediately extendable to the determination of self- and remote-influence functions, the latter corresponding to the effect of a field's production on the pressure of a neighboring reservoir situated on the same aquifer. We are currently evaluating the LP approach to this interference problem.

SUMMARY AND CONCLUSIONS

1. The problem of determining the influence function which best reproduces field history and rigorously satisfies smoothness constraints is shown to be solvable by the linear programming technique. This method is very practical because of the availability of linear programming codes for nearly all digital computers.

2. A theorem is stated and proven which validates influence function smoothness constraints previously accepted on physical intuition and establishes additional constraints previously unrecognized.

3. Application of the method to the unsmoothed data from three reservoirs resulted in realistic influence functions (i.e., not straight lines, which are allowed by the method) which gave reservoir performance predictions of engineering accuracy.

4. Further usage of the method will evaluate the generality of our experience that smoothness constraints on the first three derivatives of the influence function are sufficient in the sense that higher-order constraints have little effect.

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NOMENCLATURE

c = compressibility of aquifer rock and water
e_w = rate of water influx, volume/time
e_{wi} = average rate e_w during time increment from $(i-1)\,\Delta t$ to $i\Delta t$
$F(t)$ = influence function
F_i = value of $F(t)$ at time $i\Delta t$
$F^k = d^kF/dt^k$
K = reservoir permeability
n = number of time increments over which field performance is matched

p = reservoir pressure
p_i = p at time $i\Delta t$
p_o = initial pressure
R = gas constant
t = time
Δt = time increment
T = absolute temperature of reservoir
V_i = reservoir pore volume at time $i\Delta t$
$X_i = F_i - F_{i-1}$
z = gas compressibility factor
μ = aquifer water viscosity
ϕ = aquifer porosity

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APPENDIX

A proof is given below for the following theorem. Let $p(x,y,z,t)$ satisfy the equation

$$\bar{\nabla} \cdot [K(x,y,z)\, \overline{\nabla p}] = \rho(x,y,z) \frac{\partial p}{\partial t} \quad \text{(A-1)}$$

in a domain G bounded by a piecewise smooth surface(s), S, comprised of the separate portions S_1, S_2, S_3.

If K is positive, K, $\partial K/\partial x$, $\partial K/\partial y$, $\partial K/\partial z$ are continuous functions of position, ρ is positive and piecewise continuous, boundary and initial conditions,

$$\begin{aligned} K\,\overline{\nabla p}\cdot \bar{n} &= e_w \text{ on } S_1, \quad \text{(A-2)}\\ &= 0 \text{ on } S_2 \quad t \geq 0\\ p &= 0 \text{ on } S_3\\ p(x,y,z,0) &= 0,\ x,y,z \text{ in } G, \quad \text{(A-3)}\end{aligned}$$

hold and $F(t)$ is defined by

$$F(t) = \int_{S_1} p(x,y,z,t)\,dS \quad \text{(A-4)}$$

then $F(t)$ satisfies conditions

$$\begin{aligned} e_w F(t) &> 0\\ e_w F^{2n-1}(t) &\geq 0\\ e_w F^{2n}(t) &\leq 0 \quad \text{(A-5)}\end{aligned}$$

for all $t > 0$ and $n = 1,2,\ldots$ In addition, $F(t)$ and all its derivatives are continuous functions of time.

The physical meaning of ρ is $\mu\,\phi\,c$ where μ and c are assumed constant and porosity may vary with position. Permeability is K. Eq. A-1 is the well-known diffusivity equation governing single-phase, Darcy flow of a slightly compressible fluid through porous media.[9] Certain second-order terms are missing from this equation as they play virtually no role in aquifer-reservoir behavior. The variable p of the theorem is velocity potential $\int^{p} \frac{dp}{\rho} - h$ where the latter p is pressure, ρ is density and h vertical position. Thus if an aquifer's outcropping is defined to mean that flow potential over the "surface" of outcrop is constant, then boundary conditions (Eq. A-2) apply to the general case of an aquifer which may be closed over part of its boundary (S_2), outcrops over another portion of its boundary (S_3) and contacts the reservoir over the remainder (S_1) of its boundary. The vector n is the outward normal to the surfaces S, and e_w is a constant. The expression $\int_{S_1} p\,dS$ is the surface integral of p over S_1 and F^k is the kth derivative of $F(t)$. The operator $\bar{\nabla}$ is divergence,

$$\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}.$$

In the following proof, detailed existence theorems for two subsidiary boundary value problems below (Eqs. A-10 and A-11) are not included but are implied on a basis of Courant and Hilbert's discussion.[11] J. R. McCord has completed a rigorous proof of the above theorem results for the one-dimensional case. A later paper will present that work as well as the present theorem proof in complete rigor.

PROOF

Green's formulas for the self-adjoint differential operator, $L(p)$,

$$L(p) \equiv \bar{\nabla}\cdot(K\bar{\nabla}p) \quad \text{(A-6)}$$

are

$$\int_G [vL(u) - uL(v)]\,d\tau = \int_S \left(Kv\frac{\partial u}{\partial n} - Ku\frac{\partial v}{\partial n}\right) dS \quad \text{(A-7)}$$

$$\int_G K(u_x^2 + u_y^2 + u_z^2)\,d\tau = -\int_G uL(u)\,d\tau + \int_S Ku\frac{\partial u}{\partial n}\,ds \quad \text{(A-8)}$$

where $d\tau$ denotes $dx\,dy\,dz$, and dS is the surface element.

Let

$$p(x,y,z,t) = r(x,y,z,t) + s(x,y,z), \quad \text{(A-9)}$$

where s satisfies

$$\begin{aligned} L(s) &= 0\\ K\bar{\nabla}s\cdot\bar{n} &= e_w \text{ on } S_1\\ &= 0 \text{ on } S_2\\ s &= 0 \text{ on } S_3 \quad \text{(A-10)}\end{aligned}$$

If r satisfies

$$L(r) = \rho \frac{\partial r}{\partial t} , \quad \text{(A-11)}$$

$$\overline{\nabla} r \cdot \overline{n} = 0 \text{ on } S_1 + S_2$$
$$r = 0 \text{ on } S_3$$
$$r(x,y,z,0) = -s, \quad \text{(A-12)}$$

then the right side of Eq. A-9 is easily seen to satisfy Eqs. A-1 through A-3.

Proceeding with the standard "separation of variable" approach, we set

$$r(x,y,z,t) = U(x,y,z)\, \theta(t), \quad \text{(A-13)}$$

and substituting this into Eq. A-11, we find that

$$\theta = e^{-\lambda t}, \quad \text{(A-14)}$$

and

$$L(U) + \rho\lambda U = 0 \quad \text{(A-15)}$$

We assume (Ref. 11, Vol. I, p. 309, Section 14; Vol. II, p. 290) a complete, orthonormal set U_i as solutions to Eq. A-15 satisfying boundary conditions A-12 and an expansion theorem; i.e., $\int_G \rho U_i U_j d\tau = 0$ if $i \neq j$, 1 if $i = j$. Actually, the orthonogality of the set follows directly from Eqs. A-7 and A-15 since the right side of Eq. A-7 is zero from the boundary condition.

The eigenvalues λ_i are all positive since Eqs. A-8 and A-15 give*

$$\int_G K(U_x^2 + U_y^2 + U_z^2)\, d\tau = -\int_G U\, L(U)\, d\tau =$$
$$\lambda \cdot \int_G U^2 \rho\, d\tau = \lambda. \quad \text{(A-16)}$$

We now have the solution for r as

$$r = \sum_{i=1}^{\infty} A_i U_i e^{-\lambda_i t}, \quad \text{(A-17)}$$

where A_i are the Fourier coefficients necessary to satisfy the initial condition A-12:

$$A_i = -\int_G s(x,y,z)\, \rho\, U_i\, d\tau. \quad \text{(A-18)}$$

Eq. A-9 now becomes

*We could include a zero eigenvalue here but it can be excluded since its contribution disappears from the final solution through imposition of initial condition (A-12).

$$p = \sum_1^{\infty} A_i U_i e^{-\lambda_i t} + s$$
$$= -\sum_1^{\infty} A_i U_i (1 - e^{-\lambda_i t}),$$

and Definition A-4 gives

$$F(t) = -\sum_1^{\infty} A_i \int_{S_1} U_i\, ds\, (1 - e^{-\lambda_i t}). \quad \text{(A-19)}$$

Use of Eqs. A-18, A-15, A-17, A-10 and A-12 gives

$$A_i = -\int_G s\, \rho\, U_i\, d\tau = \frac{1}{\lambda_i}\int_G s\, L(U_i)\, d\tau$$
$$= \frac{1}{\lambda_i}\int_G U_i\, L(s)\, d\tau + \int_S K\, s\, \overline{\nabla} U_i \cdot \overline{n} dS -$$
$$\int_S K\, U_i\, \overline{\nabla s} \cdot \overline{n} dS = -\frac{e_w}{\lambda_i}\int_{S_1} U_i\, dS \quad \text{(A-20)}$$

Thus, finally, Eq. A-19 becomes

$$F(t) = e_w \sum_1^{\infty} \frac{1}{\lambda_i} \left(\int_{S_1} U_i\, dS\right)^2 (1 - e^{-\lambda_i t}) \quad \text{(A-21)}$$

The properties to be proved, Eq. A-5, follow immediately from A-21 since all λ_i are positive. The continuity of $F(t)$ and all its derivatives follows from the form of Eq. A-21.

The proof of the theorem for a closed aquifer, where boundary conditions (Eq. A-2) are

$$K\overline{\nabla} p \cdot \overline{n} = e_w \text{ on } S_1$$
$$K\overline{\nabla} p \cdot \overline{n} = 0 \text{ on } S\text{-}S_1, \quad \text{(A-22)}$$

is very similar to the above proof. Eq. A-9 is replaced by

$$p = r(x,y,z,t) + \gamma t + s(x,y,z) \quad \text{(A-23)}$$

where $L(s)$ is now $\rho\gamma$ and the constant γ is easily found as $e_w S_1 / V$, with $V = \int_G \rho\, d\tau$. The final result is

$$F(t) = e_w \sum_1^{\infty} \frac{1}{\lambda_i} \left(\left(\int_{S_1} U_i\, dS\right)^2 (1 e^{-\lambda_i t}) \right)$$
$$\frac{e_w S_1^2}{V} t, \quad \text{(A-24)}$$

from which the properties A-5 follow. ***